



Integral manifolds of singularly perturbed systems with application to rigid-link flexible-joint multibody systems

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Abstract

In this paper, we first review results of integral manifolds of singularly perturbed non-linear differential equations. We then outline the basic elements of the integral manifold method in the context of control system design, namely, the existence of an integral manifold, its attractivity, and stability of the equilibrium while the dynamics are restricted to the manifold. Toward this end, we use the composite Lyapunov method and propose a new exponential stability result which gives, as a by-product, an explicit range of the small parameter for which exponential stability is guaranteed. The results are applied to the control problem of multibody systems with rigid links and flexible joints in which the inverse of joint stiffness plays the role of the small parameter. The proposed controller is a composite control law that consists of a fast component, as well as a slow component that was designed based on the integral manifold approach. We show that the proposed composite controller has the following properties: (i) it enables the exact characterization and computation of an integral manifold, (ii) it makes the manifold exponentially attractive, and (iii) it forces the dynamics of the reduced flexible system on the integral manifold to coincide with the dynamics of the corresponding rigid system (i.e. the one obtained by making stiffness very large) implying that any control law that stabilizes the rigid system would stabilize the dynamics of the flexible system on the manifold. We finally present a detailed stability analysis and give an explicit range of the joint stiffness, in terms of system parameters and controller gains, for which the established exponential stability is guaranteed. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Integral manifolds; Singularly perturbed non-linear differential equations; Lyapunov stability; Multibody systems; Flexible joints; Composite control

Nomenclature

\mathbb{R}_+
 \mathbb{R}^n

set of non-negative real numbers
usual n -dimensional vector space over \mathbb{R}

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$\ x\ = \{\sum_{i=1}^n x_i^2\}^{1/2}$	Euclidean norm in \mathbb{R}^n
$\mathbb{R}^{n \times n}$	set of all $n \times n$ matrices with real elements
$\ A\ = \{\lambda_{\max}(A^T A)\}^{1/2}$	Induced matrix norm of $A \in \mathbb{R}^{n \times n}$ corresponding to the Euclidean vector norm where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$
$B_z \subset \mathbb{R}^p$	compact ball centered at $z = 0, z \in \mathbb{R}^p, p$ integer, that is, $B_z = \{z \in \mathbb{R}^p : \ z\ \leq \rho_z, \rho_z \text{ positive real}\} \subset \mathbb{R}^p$

1. Introduction

We recall the definition of an integral manifold [1] for the differential equation

$$\dot{X} = N(t, X), \tag{1}$$

where $X, N \in \mathbb{R}^n$. The set $\mathcal{M} \subset \mathbb{R} \times \mathbb{R}^n$ is said to be an *integral manifold* (invariant manifold) if for $(t_0, X_0) \in \mathcal{M}$, the solution $(t, X(t)), X(t_0) = X_0$, is in \mathcal{M} for $t \in \mathbb{R}$. If $(t, X(t)) \in \mathcal{M}$ for only a finite interval of time, then \mathcal{M} is said to be a *local integral manifold*.

In order to illustrate the basic ideas of exploiting the integral manifold method in the analysis and control of dynamical systems, consider the special case in which Eq. (1) is autonomous and $X = [x, w]^T \in \mathbb{R}^2$ so that

$$\begin{aligned} \dot{x} &= f(x, w, u_1(x, w)), \\ \dot{w} &= g(x, w, u_2(x, w)), \end{aligned}$$

where $x, f, w, g \in \mathbb{R}$, and $u = [u_1, u_2]^T$ is a control input of appropriate dimension.

Fig. 1 illustrates the three basic issues of integral manifolds of systems described by the above ordinary differential equation in the context of control system design. First, the *existence* of an integral manifold \mathcal{M} (recall definition above) needs to be established so that if the initial states start on \mathcal{M} , the trajectory of the system remains on \mathcal{M} thereafter. Second, when restricted to the integral manifold \mathcal{M} , the dynamics of the system should insure *stability* of the equilibrium. Third, the integral manifold \mathcal{M} should be *attractive* so that if the initial conditions are off \mathcal{M} as shown in Fig. 1, the solution trajectory asymptotically converges to \mathcal{M} . In a control system design context, the challenge is therefore to devise an appropriate control law $u(x, w)$ that insures the existence of an attractive integral manifold, and furthermore, insures stability of the system when the dynamics are restricted to the integral manifold. Note that the fundamental advantage of an integral manifold approach to control system design is that once an attractive integral manifold is designed for a dynamical system, the stability problem of the original system reduces to a stability problem of a lower dimensional system on the manifold which is typically much easier to deal with as we will illustrate with the class of mechanical systems considered in this paper.

In the first part of the paper (Section 2), we present a summary of results from the literature on integral manifolds of singularly perturbed non-linear differential equations which establish the existence and mathematical properties of integral manifolds for this class of dynamical system. Since typical stability statements in the mathematical literature about singularly perturbed systems possessing integral manifolds are not particularly appealing in the context of control system design, we use the composite Lyapunov method and propose a new exponential stability result which gives, as a by-product, an explicit range of the small parameter for which the shown stability is guaranteed. This proves to be a very useful engineering result for the following reasons. First Lyapunov method is a powerful tool for combined controller design and stability analysis. Second, because the explicit range of the small parameter of the singularly perturbed system is typically expressed in terms of plant parameters and controller gains, such

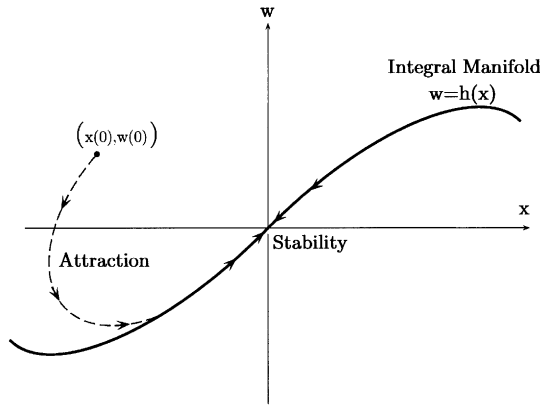


Fig. 1. Integral manifold of an autonomous system.

range could be maximized by plant parameter re-design as well as by proper choice of controller gain parameters.

In the second part of the paper (Section 3), the results are applied to the control of multibody mechanical systems with rigid links and flexible joints in which the inverse of joint stiffness plays the role of the small parameter. The proposed controller is a composite control law that consists of two components. The first is a fast component that will dampen fast oscillations due to joint flexibility and will have the interpretation of making an integral manifold attractive. The second component is a slow component that is designed based on the integral manifold approach. We show that the proposed composite controller achieves the following goals: first, it enables the exact characterization and computation of the integral manifold; this is a unique situation because it is generally impossible to compute the expression of the integral manifold as it is equivalent to solving the original non-linear singularly perturbed differential equation system. Second, the controller, specifically the fast component, makes the manifold exponentially attractive. Third, the controller forces the dynamics of the reduced flexible system, while on the integral manifold, to coincide with the dynamics of the corresponding rigid system (the one obtained when stiffness is made very large). This in particular implies that all control laws, available from the control literature, that stabilize the rigid system would also stabilize the dynamics of the flexible system on the manifold. Hence the slow controller consists of a first component designed based on the rigid system and two more components achieving the above mentioned goals. It is noted that only link positions and velocities and rotor velocities are used for feedback. Finally, we give a detailed stability analysis using the composite Lyapunov approach previously proposed, and give an explicit range of the joint stiffness, in terms of the system parameters and controller gains, for which the established exponential stability is guaranteed.

In the last part of the paper in Section 4, conclusions are drawn.

2. Integral manifolds of non-linear singularly perturbed systems

Consider the following system of differential equations:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, \mathbf{y}, \varepsilon), \tag{2}$$

$$\varepsilon \dot{\mathbf{y}} = g(t, \mathbf{x}, \mathbf{y}, \varepsilon), \tag{3}$$

where $\mathbf{x}, f \in \mathbb{R}^n$, $\mathbf{y}, g \in \mathbb{R}^m$, $t \in \mathbb{R}$, and ε is a small parameter.

2.1. Existence and properties of integral manifold

In this section, we group some results on integral manifolds of singularly perturbed non-linear differential equations and then outline and discuss the approach of the proof.

Lemma 2.1 (Existence and properties of integral manifold). *Suppose the following hypotheses hold:*

- \mathcal{A}_1 : The algebraic equation $g(t, \mathbf{x}, \mathbf{y}, 0) = 0$ has an isolated solution $\mathbf{y} = h_0(t, \mathbf{x})$, $\forall t \in \mathbb{R}$, $\forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}$.
- \mathcal{A}_2 : The functions f , g , and h_0 are twice continuously differentiable ($\in C^2$) $\forall t \in \mathbb{R}$, $\forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}$, $\forall \varepsilon \in [0, \varepsilon_0)$, and for $\|\mathbf{y} - h_0(t, \mathbf{x})\| \leq \bar{\rho}_y$ where ε_0 and $\bar{\rho}_y$ are positive real constants.
- \mathcal{A}_3 : The eigenvalues $\lambda_i = \lambda_i(t, \mathbf{x})$, $i = 1, 2, \dots, m$, of the matrix $Z(t, \mathbf{x}) := (\partial g / \partial \mathbf{y})(t, \mathbf{x}, h_0(t, \mathbf{x}), 0)$ satisfy the inequality

$$\operatorname{Re}[\lambda_i] \leq -2\beta < 0 \quad \forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}. \quad (4)$$

Then, there exists $\varepsilon_1 \leq \varepsilon_0$ such that $\forall \varepsilon \in [0, \varepsilon_1)$, the singularly perturbed system (2),(3) has an m -dimensional local integral manifold

$$\mathcal{M}_\varepsilon: \mathbf{y} = h_0(t, \mathbf{x}) + H(t, \mathbf{x}, \varepsilon) := h(t, \mathbf{x}, \varepsilon), \quad (5)$$

where $h(t, \mathbf{x}, \varepsilon)$ is defined for all $\mathbf{x} \in \mathbf{B}_{\mathbf{x}}$ and $\varepsilon \leq \varepsilon_1$, is continuously differentiable ($\in C^1$) and $H(t, \mathbf{x}, \varepsilon)$ satisfies the inequalities

$$\|H(t, \mathbf{x}, \varepsilon)\| \leq \rho_1(\varepsilon),$$

$$\|H(t, \mathbf{x}', \varepsilon) - H(t, \mathbf{x}'', \varepsilon)\| \leq \rho_2(\varepsilon)\|\mathbf{x}' - \mathbf{x}''\|,$$

where $\rho_1(\varepsilon) \rightarrow 0$ and $\rho_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The function $h(t, \mathbf{x}, \varepsilon) \in C^1$ satisfies the so-called manifold equation

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial \mathbf{x}} f(t, \mathbf{x}, h, \varepsilon) = g(t, \mathbf{x}, h, \varepsilon), \quad (6)$$

which is obtained by substituting \mathbf{y} by h in Eq. (3). On this manifold, the flow of systems (2), (3) is governed by the n -dimensional reduced system

$$\dot{\mathbf{x}} = f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon). \quad (7)$$

Furthermore, if for $\mathbf{x} \in \mathbf{B}_{\mathbf{x}}$ and p integer we have $f(t, \mathbf{x}, \mathbf{y}, \varepsilon) \in C^{p+1}$, $g(t, \mathbf{x}, \mathbf{y}, \varepsilon) \in C^{p+2}$, and $h_0(t, \mathbf{x}) \in C^{p+2}$, then $h \in C^p$. \square

Proof (DISCUSSION AND OUTLINE). As surveyed in Knobloch and Aulback [2], the technique of applying invariant manifold methods to singularly perturbed non-linear differential equations was first recognized by Zadiraka in 1957 where in his first paper [3] he showed the existence of a local integral manifold. In a later paper [4], he proved in 1965 the existence of a global integral manifold. A major assumption in Zadiraka's work is that \mathcal{A}_3 of Lemma 2.1 is satisfied. Baris [5] weakened assumption \mathcal{A}_3 by allowing some of the eigenvalues to have strictly positive real parts. Both Zadiraka and Baris showed the existence of the integral manifold using the contraction mapping theorem (also sometimes known as the Banach fixed point theorem). Geometric methods were used by Fenichel [6] in 1979 to study the autonomous case and later in 1981, Henry [7] proved the corresponding result for a class of non-autonomous differential equations. A common feature in the invariant manifold method is that the integral manifolds were constructed by extrapolating the degenerate manifold (i.e. the manifold obtained when $\varepsilon = 0$). More techniques and results are also available for analyzing and constructing integral manifolds (see for example [2], Knobloch [2,8–10]).

Since in this paper we are interested in the design and analysis of stable systems, assumption \mathcal{A}_3 is not restrictive. On the contrary, it is desirable for the stability analysis as we will see later in the paper. A typical assumption needed in the proofs of the existence of integral manifolds, including Zadiraka’s work [4], is that the function $f(t, \mathbf{x}, \mathbf{y}, \varepsilon)$ be bounded for $t \in \mathbb{R}$, $\varepsilon \leq \varepsilon_0$, $\|\mathbf{y}\| \leq \rho_y$, and that \mathbf{x} belong to an open unbounded subset of the Euclidean space \mathbb{R}^n usually taken as the whole \mathbb{R}^n . It can easily be checked that such assumption does not hold globally in \mathbf{x} for systems like second (and higher)-order linear systems and mechanical systems of the type considered in this paper. Thus, in order to carry the proof of the existence of integral manifold, we could borrow ideas originally used by Hausrath [11] in his work on stability of neutral functional differential equations and later by Carr [12] to prove center manifolds for autonomous non-linear systems. Basically, the idea consists of choosing a compact ball $\mathbf{B}_x \subset \mathbb{R}^n$, and considering a modified system which is identical to the original system when $\mathbf{x} \in \mathbf{B}_x$ but has the global (in \mathbf{x}) properties required. We prove the existence of a global (in t) integral manifold for the modified system, and consequently the existence of a local (in t) integral manifold for the original system. That is, we show that if \mathbf{x} and \mathbf{y} start initially on the integral manifold, then they remain on the manifold as long as \mathbf{x} is still in the compact set \mathbf{B}_x . In a later theorem when we consider the stability problem we will present conditions under which \mathbf{x} will never leave the compact set \mathbf{B}_x , and hence the existence of a global (in t) integral manifold. Note that the size of \mathbf{B}_x can be chosen to have any fixed size as long as it is compact. A detailed proof of Lemma 2.1 is not included in this paper and can be found in [13]. \square

2.2. Stability analysis

Typical stability statements include for example (see [1,7])

Fact 2.1. *Under the same assumptions of Lemma 2.1, there exists $\varepsilon^* \leq \varepsilon_1$ such that $\forall \varepsilon \in [0, \varepsilon^*]$, and for any solution $\mathbf{x}(t)$, $\mathbf{y}(t)$, $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{y}(t_0) = \mathbf{y}_0$, of systems (2), (3) with sufficiently small $\|\mathbf{y}_0 - h(t, \mathbf{x}_0, \varepsilon)\|$, there is a solution $\mathbf{m}(t)$, $\mathbf{m}(t_0) = \mathbf{m}_0$ of Eq. (7) satisfying*

$$\mathbf{x}(t) = \mathbf{m}(t) + \phi_1(t),$$

$$\mathbf{y}(t) = h(t, \mathbf{m}(t), \varepsilon) + \phi_2(t),$$

where $\phi_i(t) = O(e^{-(\beta/\varepsilon)(t-t_0)})$ as $(t-t_0) \rightarrow \infty$, $i = 1, 2$. Furthermore, if the equilibrium of Eq. (7) is stable (asymptotically stable, unstable), then the equilibrium of systems (2), (3) is stable (asymptotically stable, unstable). \square

Our stability analysis below differs from above asymptotic statements in two major points. First, we use the composite Lyapunov method [14] to prove stability of the equilibrium of the singularly perturbed systems (2), (3). Such method is more appealing from an engineering point of view since Lyapunov methodology is a useful tool for combined design of control laws and stability analysis. Second, we will get as a by-product an explicit expression for an upper bound ε^* of ε (in terms of system parameters and controller gains) up to which established stability properties will be maintained. This is an extremely useful result since such method gives us a mechanism through which system parameters and controller gains could be chosen to allow for larger ε^* . Note that ε is usually directly related to a physical parameter of the dynamical system, and consequently maximizing it while maintaining stability properties is usually a very desirable result. In what follows in this section, we assume that \mathcal{A}_1 – \mathcal{A}_3 of Lemma 2.1 are satisfied, and hence there exists a local integral manifold \mathcal{M}_ε as given in Lemma 2.1.

We first introduce the variable

$$\boldsymbol{\eta} := \mathbf{y} - h(t, \mathbf{x}, \varepsilon).$$

Consider the following system which represents systems (2), (3) on and off the manifold \mathcal{M}_ε :

$$\dot{\mathbf{x}} = F(t, \mathbf{x}, \eta, \varepsilon), \quad (8)$$

$$\varepsilon \dot{\eta} = Z(t, \mathbf{x})\eta + G(t, \mathbf{x}, \eta, \varepsilon), \quad (9)$$

where

$$F(t, \mathbf{x}, \eta, \varepsilon) := f(t, \mathbf{x}, \eta + h(t, \mathbf{x}, \varepsilon), \varepsilon),$$

$$\begin{aligned} G(t, \mathbf{x}, \eta, \varepsilon) &:= -Z(t, \mathbf{x})\eta + \varepsilon \dot{\mathbf{y}} - \varepsilon \dot{h}(t, \mathbf{x}, \varepsilon) = -Z(t, \mathbf{x})\eta + g(t, \mathbf{x}, \eta + h, \varepsilon) - \varepsilon \frac{\partial h}{\partial t} - \varepsilon \frac{\partial h}{\partial \mathbf{x}} f(t, \mathbf{x}, \eta + h, \varepsilon) \\ &= -Z(t, \mathbf{x})\eta + g(t, \mathbf{x}, \eta + h, \varepsilon) - \varepsilon \left\{ \frac{\partial h}{\partial t} + \frac{\partial h}{\partial \mathbf{x}} f(t, \mathbf{x}, h, \varepsilon) \right\} - \varepsilon \frac{\partial h}{\partial \mathbf{x}} \{f(t, \mathbf{x}, \eta + h, \varepsilon) - f(t, \mathbf{x}, h, \varepsilon)\} \\ &= -Z(t, \mathbf{x})\eta + g(t, \mathbf{x}, \eta + h(t, \mathbf{x}, \varepsilon), \varepsilon) - g(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon) \\ &\quad - \varepsilon \frac{\partial h(t, \mathbf{x}, \varepsilon)}{\partial \mathbf{x}} \{f(t, \mathbf{x}, \eta + h(t, \mathbf{x}, \varepsilon), \varepsilon) - f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon)\}. \end{aligned} \quad (10)$$

Note that in Eq. (10), we used Eq. (6). In what follows, we let the compact set \mathbf{B}_η denote the domain of $\eta = \mathbf{y} - h(t, \mathbf{x}, \varepsilon)$ such that $\mathbf{y} \in \mathbf{B}_y$ and $\mathbf{x} \in \mathbf{B}_x$. Hence systems (8), (9) is defined on $\mathbb{R} \times \mathbf{B}_x \times \mathbf{B}_\eta \times [0, \varepsilon_1]$ where ε_1 is given in Lemma 2.1.

Remark 2.1. Since the functions f , g (by hypothesis), and h (from Lemma 2.1) are continuously differentiable, we conclude that they are locally Lipschitz and their derivatives with respect to all variables are bounded when \mathbf{x} is restricted to \mathbf{B}_x .

We have the following properties.

Property 2.1. $\forall \mathbf{x}', \mathbf{x}'' \in \mathbf{B}_x, \forall \eta', \eta'' \in \mathbf{B}_\eta, \forall t \in \mathbb{R}, \forall \varepsilon \in [0, \varepsilon^1]$, there exists a positive constant c_1 such that

$$\|F(t, \mathbf{x}'', \eta'', \varepsilon) - F(t, \mathbf{x}', \eta', \varepsilon)\| \leq c_1(\|\mathbf{x}'' - \mathbf{x}'\| + \|\eta'' - \eta'\|).$$

Proof. Follows from the fact that $F(t, \mathbf{x}, \eta, \varepsilon) = f(t, \mathbf{x}, \eta + h(t, \mathbf{x}, \varepsilon), \varepsilon)$ and from Remark 2.1 that both f and h are Lipschitz. \square

Property 2.2. $\forall \mathbf{x} \in \mathbf{B}_x, \forall \eta \in \mathbf{B}_\eta, \forall t \in \mathbb{R}, \forall \varepsilon \in [0, \varepsilon^1]$, there exist positive constants c_2 and c_3 such that

$$\|G(t, \mathbf{x}, \eta, \varepsilon)\| \leq (c_2 + c_3\varepsilon)\|\eta\|. \quad (11)$$

Proof. Using Eq. (10)

$$\begin{aligned} G(t, \mathbf{x}, \eta, \varepsilon) &= -Z(t, \mathbf{x})\eta + g(t, \mathbf{x}, \eta + h(t, \mathbf{x}, \varepsilon), \varepsilon) - g(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon) \\ &\quad - \varepsilon \frac{\partial h(t, \mathbf{x}, \varepsilon)}{\partial \mathbf{x}} \{f(t, \mathbf{x}, \eta + h(t, \mathbf{x}, \varepsilon), \varepsilon) - f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon)\} \end{aligned}$$

$$\begin{aligned} &= -Z(t, \mathbf{x})\eta + \frac{\partial g}{\partial \mathbf{y}}(t, \mathbf{x}, \bar{v}_1\eta + h(t, \mathbf{x}, \varepsilon), \varepsilon)\eta - \varepsilon \frac{\partial h(t, \mathbf{x}, \varepsilon)}{\partial \mathbf{x}} \frac{\partial f}{\partial \mathbf{y}}(t, \mathbf{x}, \bar{v}_2\eta + h(t, \mathbf{x}, \varepsilon), \varepsilon)\eta \\ &= \left\{ -Z(t, \mathbf{x}) + \frac{\partial g}{\partial \mathbf{y}}(t, \mathbf{x}, \bar{v}_1\eta + h(t, \mathbf{x}, \varepsilon), \varepsilon) - \varepsilon \frac{\partial h(t, \mathbf{x}, \varepsilon)}{\partial \mathbf{x}}(t, \mathbf{x}, \bar{v}_2\eta + h(t, \mathbf{x}, \varepsilon), \varepsilon) \right\} \eta, \end{aligned}$$

where $\bar{v}_1, \bar{v}_2 \in (0, 1)$. Hence, $\forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}, \forall \eta \in \mathbf{B}_{\eta}, \forall t \in \mathbb{R}, \forall \varepsilon \in [0, \varepsilon_1]$, and using Remark 2.1, there exist positive constants c_2 and c_3 such that Eq. (11) is satisfied.

We now define the following auxiliary system:

$$\varepsilon \dot{\eta} = Z(t, \mathbf{x})\eta, \tag{12}$$

where Z depends explicitly on t and \mathbf{x} . We first recall the properties of Z .

Property 2.3. The eigenvalues $\lambda_i = \lambda_i(t, \mathbf{x}), i = 1, 2, \dots, m$, of $Z(t, \mathbf{x})$ satisfy inequality (4) $\forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}$.

Property 2.4. $\forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}$, there exists positive constants ρ_{z1}, ρ_{z2} such that

$$\|Z(t, \mathbf{x})\| < \rho_{z1}, \tag{13}$$

$$\|\dot{Z}(t, \mathbf{x}, \mathbf{y}, \varepsilon)\| = \left\| \frac{\partial Z}{\partial t} + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial Z_{ij}}{\partial \mathbf{x}} F(t, \mathbf{x}, \eta, \varepsilon) \right\| < \rho_{z2}. \tag{14}$$

We now have the following lemma (see [15], p. 203).

Lemma 2.2. Since $Z(t, \mathbf{x})$ satisfies Property 2.3 and inequality (13) in Property 2.4, then there exist positive constants τ_{01} and τ_{02} such that

$$\|e^{Z(t, \mathbf{x})s}\| \leq \tau_{01} e^{-\tau_{02}s} \quad \forall s > 0, \forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}.$$

Let $P(t, \mathbf{x})$ be the solution of the following algebraic Lyapunov equation parameterized in t and \mathbf{x} :

$$P(t, \mathbf{x})Z(t, \mathbf{x}) + Z^T(t, \mathbf{x})P(t, \mathbf{x}) = -I_m. \tag{15}$$

We have the following lemma (see [15], p. 244):

Lemma 2.3. Let

$$P(t, \mathbf{x}) = \int_0^\infty [e^{Z(t, \mathbf{x})s}]^T [e^{Z(t, \mathbf{x})s}] ds,$$

where $Z(t, \mathbf{x})$ satisfies Properties 2.3 and 2.4. Then $\forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_{\mathbf{x}}$

- (i) Eq. (15) is satisfied.
- (ii)

$$\frac{1}{2\rho_{z1}} \eta^T \eta \leq \eta^T P(t, \mathbf{x}) \eta \leq \frac{q_{01}^2}{2\tau_{02}} \eta^T \eta, \tag{16}$$

- (iii)

$$\dot{P}(t, \mathbf{x}, \eta, \varepsilon) = \int_0^\infty [e^{Z(t, \mathbf{x})s}]^T [P(t, \mathbf{x})\dot{Z}(t, \mathbf{x}) + \dot{Z}^T(t, \mathbf{x})P(t, \mathbf{x})] [e^{Z(t, \mathbf{x})s}] ds,$$

(iv)

$$\|\dot{P}(t, \mathbf{x}, \eta, \varepsilon)\| \leq \frac{\rho_{z2}\tau_{01}^4}{2\tau_{02}^2} =: \tau_1. \quad (17)$$

Hence, Properties 2.3 and 2.4 guarantee that $\forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_x$, Eq. (15) has a unique positive-definite solution $P(t, \mathbf{x})$ with properties (i)–(iv).

Let us use the following quadratic Lyapunov function candidate for the auxiliary system (12):

$$W(t, \mathbf{x}, \eta) = \eta^T P(t, \mathbf{x}) \eta. \quad (18)$$

Then, using Lemma 2.3, the time derivative of W along the solution trajectories of the auxiliary system (12) is given by

$$\begin{aligned} \dot{W} &= [\nabla_{\eta} W(t, \mathbf{x}, \eta)]^T \left(\frac{1}{\varepsilon} Z(t, \mathbf{x}) \eta \right) + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon) \eta \\ &= \frac{1}{\varepsilon} \eta^T (P(t, \mathbf{x}) Z(t, \mathbf{x}) + Z^T(t, \mathbf{x}) P(t, \mathbf{x})) \eta + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon) \eta \leq \left(-\frac{1}{\varepsilon} + \tau_1 \right) \|\eta\|^2. \end{aligned}$$

The reduced system is given by (see Eq. (7))

$$\dot{\mathbf{x}} = F(t, \mathbf{x}, \eta = 0, \varepsilon) = f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon). \quad (19)$$

Suppose that for the reduced system (19), there exists a Lyapunov function $V(t, \mathbf{x})$ that satisfies

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}} F(t, \mathbf{x}, 0, \varepsilon) \leq -\tau_2 \|\mathbf{x}\|^2, \quad (20)$$

$$\|\nabla_{\mathbf{x}} V(t, \mathbf{x})\| \leq \frac{q_3}{c_1} \|\mathbf{x}\| \quad (21)$$

for some positive constants τ_2 and τ_3 . Define

$$\Omega_v = \{\mathbf{x} \in B_x : V(t, \mathbf{x}) \leq c_v\}, \quad (22)$$

where c_v is the largest constant such that Ω_v is contained in B_x . Similarly, let

$$\Omega_w = \{\mathbf{x} \in B_x, \eta \in B_{\eta} : W(t, \mathbf{x}, \eta) \leq c_w\} \quad (23)$$

where c_w is the largest constant such that Ω_w is contained in $B_x \times B_{\eta}$.

We have the following result:

Theorem 2.1 (Stability analysis). *Assume the following assumptions hold:*

- \mathcal{B}_1 : Assumptions \mathcal{A}_1 – \mathcal{A}_3 of Lemma 2.1 are satisfied.
- \mathcal{B}_2 : For the reduced system (19), there exists a Lyapunov function $V(t, \mathbf{x})$ that satisfies Eqs. (20) and (21).

Then there exists an upper bound of ε given by

$$\varepsilon_d = \frac{\tau_2(1 - \tau_5)}{\tau_2(\tau_1 - \tau_4) + (d_1/4d_2)\tau_3^2}$$

such that $\forall \varepsilon \in [0, \varepsilon_d), \forall (\mathbf{x}(t_0), \eta(t_0)) \in \Omega_{\mathcal{V}}$ where

$$\Omega_{\mathcal{V}} = \{ \mathbf{x} \in \mathbf{B}_{\mathbf{x}}, \eta \in \mathbf{B}_{\eta} : \mathcal{V}(t, \mathbf{x}, \eta) \leq \min [d_1 c_v, d_2 c_w] \},$$

we have

$$\lim_{t \rightarrow \infty} \eta(t) = 0 \quad (\text{i.e. } \mathbf{y}(t) \rightarrow h(t, \mathbf{x}, \varepsilon) \text{ as } t \rightarrow \infty), \tag{24}$$

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = 0. \tag{25}$$

d_1, d_2 are positive real constants, $c_v, c_w, \tau_1, \tau_2, \tau_3, \tau_4$ and τ_5 are given by Eqs. (22), (23) (17), (20), (21), (27) and (28), respectively.

If in addition, there exist constants s_1, s_2 such that

$$s_1 \|\mathbf{x}\|^2 \leq V(t, \mathbf{x}) \leq s_2 \|\mathbf{x}\|^2, \tag{26}$$

then the equilibrium $\mathbf{x} = 0, \eta = 0$ (i.e. $y = h$) of systems (8), (9) is exponentially stable.

Proof of Theorem 2.1. If \mathcal{B}_1 (i.e. $\mathcal{A}_1 - \mathcal{A}_3$) are satisfied, then from Lemma 2.1 there exists an $\varepsilon_1 > 0$ such that systems (2), (3) has a local integral manifold \mathcal{M}_ε , for $\mathbf{x} \in \mathbf{B}_{\mathbf{x}}, \mathbf{y} \in \mathbf{B}_{\mathbf{y}}, \varepsilon \in [0, \varepsilon_1)$ such that systems (5), (6) are satisfied. In addition, we can define the Lyapunov function $W(t, \mathbf{x}, \eta)$ given by Eq. (18). Keeping in mind assumption \mathcal{B}_2 , we now consider the following composite Lyapunov function candidate:

$$\mathcal{V}(t, \mathbf{x}, \eta) = d_1 V(t, \mathbf{x}) + d_2 W(t, \mathbf{x}, \eta), \quad d_1, d_2 > 0,$$

for the original singularly perturbed systems (8), (9). $V(t, \mathbf{x})$ is by assumption a locally positive definite function and decrescent. $W(t, \mathbf{x}, \eta)$ is also a locally positive-definite function and decrescent as it is clear from Lemma 2.3. Therefore, $\mathcal{V}(t, \mathbf{x}, \eta)$ is locally positive definite and decrescent. The time derivative of \mathcal{V} along the solution trajectories of systems (8), (9) is

$$\begin{aligned} \dot{\mathcal{V}}(t, \mathbf{x}, \eta, \varepsilon) &= d_1 \left\{ \frac{\partial V}{\partial t} + [\nabla_{\mathbf{x}} V(t, \mathbf{x})]^T F(t, \mathbf{x}, \eta, \varepsilon) \right\} + d_2 \left\{ [\nabla_{\eta} W(t, \mathbf{x}, \eta)]^T \frac{1}{\varepsilon} (Z(t, \mathbf{x}) \eta \right. \\ &\quad \left. + G(t, \mathbf{x}, \eta, \varepsilon)) + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon) \eta \right\} \\ &= d_1 \left\{ \frac{\partial V}{\partial t} + [\nabla_{\mathbf{x}} V(t, \mathbf{x})]^T F(t, \mathbf{x}, 0, \varepsilon) \right\} + d_1 \left\{ [\nabla_{\mathbf{x}} V(t, \mathbf{x})]^T (F(t, \mathbf{x}, \eta, \varepsilon) - F(t, \mathbf{x}, 0, \varepsilon)) \right\} \\ &\quad + d_2 \left\{ \frac{1}{\varepsilon} [\nabla_{\eta} W(t, \mathbf{x})]^T Z(t, \mathbf{x}) \eta + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon) \eta \right\} + \frac{d_2}{\varepsilon} \left\{ [\nabla_{\eta} W(t, \mathbf{x})]^T G(t, \mathbf{x}, \eta, \varepsilon) \right\} \\ &\leq -d_1 \tau_2 \|\mathbf{x}\|^2 + d_1 \tau_3 \|\mathbf{x}\| \|\eta\| + \left(d_2 \tau_1 - \frac{d_2}{\varepsilon} \right) \|\eta\|^2 + \frac{d_2}{\varepsilon} \{ 2 \|\eta\| \|P\| (\varepsilon c_3 + c_2) \|\eta\| \} \\ &\leq -d_1 \tau_2 \|\mathbf{x}\|^2 + d_1 \tau_3 \|\mathbf{x}\| \|\eta\| + \left(d_2 \tau_1 - \frac{d_2}{\varepsilon} \right) \|\eta\|^2 + d_2 \tau_4 \|\eta\|^2 + \frac{d_2}{\varepsilon} \tau_5 \|\eta\|^2 \\ &\leq -d_1 \tau_2 \|\mathbf{x}\|^2 + d_1 \tau_3 \|\mathbf{x}\| \|\eta\| + \left(d_2 (\tau_1 + \tau_4) + \frac{d_2}{\varepsilon} (\tau_5 - 1) \right) \|\eta\|^2 = -[\|\mathbf{x}\| \|\eta\|] P_d \begin{bmatrix} \|\mathbf{x}\| \\ \|\eta\| \end{bmatrix}, \end{aligned}$$

where

$$P_d := \begin{bmatrix} d_1 \tau_2 & -\frac{d_1 \tau_3}{2} \\ -\frac{d_1 \tau_3}{2} & -d_2(\tau_1 + \tau_4) + \frac{d_2}{\varepsilon}(1 - \tau_5) \end{bmatrix}.$$

Note that

$$\tau_4 := 2c_3 \|P(t, \mathbf{x})\|, \quad (27)$$

$$\tau_5 := 2c_2 \|P(t, \mathbf{x})\|, \quad (28)$$

$\dot{\mathcal{V}}$ is negative definite when P_d is positive definite, that is,

$$-d_1 \tau_2 d_2 (\tau_1 + \tau_4) + \frac{1}{\varepsilon} d_2 d_1 \tau_2 (1 - \tau_5) - \frac{d_1^2}{4} \tau_3^2 > 0$$

or, equivalently,

$$\varepsilon < \frac{\tau_2(1 - \tau_5)}{\tau_2(\tau_1 + \tau_4) + (d_1/4d_2)\tau_3^2} =: \varepsilon_d.$$

Given c_v and c_w from Eqs. (22) and (23), respectively, consider the set

$$\Omega_{\mathcal{V}} = \{\mathbf{x} \in \mathbf{B}_x, \eta \in \mathbf{B}_\eta : \mathcal{V}(t, \mathbf{x}, \eta) = d_1 V(t, \mathbf{x}) + d_2 W(t, \mathbf{x}, \eta) \leq \min[d_1 c_v, d_2 c_w]\}.$$

Then, $\forall (\mathbf{x}, \eta) \in \Omega_{\mathcal{V}}$, and $\forall \varepsilon \in [0, \varepsilon_d]$, $\dot{\mathcal{V}} < 0 \forall t \in \mathbb{R}$. and hence Eqs. (25) and (24) follow. If in addition Eq. (26) is satisfied (i.e. the equilibrium of the reduced system (19) is exponentially stable), then using also Eq. (16), we conclude that there exist positive constants a_1 and a_2 such that

$$a_1 \left\| \begin{matrix} \mathbf{x} \\ \eta \end{matrix} \right\|^2 \leq \mathcal{V}(t, \mathbf{x}, \eta) \leq a_2 \left\| \begin{matrix} \mathbf{x} \\ \eta \end{matrix} \right\|^2.$$

Moreover,

$$\dot{\mathcal{V}}(t, \mathbf{x}, \eta) \leq -\lambda_{\min}[P_d] \left\| \begin{matrix} \mathbf{x} \\ \eta \end{matrix} \right\|^2.$$

Hence we conclude that the zero equilibrium $\mathbf{x} = 0, \eta = 0$ of systems (8), (9) is exponentially stable. \square

3. Application to the control of rigid-link flexible-joint multibody systems

In this section, we apply the tools developed earlier to multibody systems with rigid links and flexible joints. A one rigid-link flexible-joint example is shown in Fig. 2. Joint flexibility is a result of motion transmission elements including gears, belts and other flexible coupling connections. Actuators are therefore elastically coupled to the rigid links through the flexible transmission elements. A model for rigid-link flexible-joint multibody systems has been derived in [16] using the following assumptions:

- The kinetic energy of the rotor is due mainly to its own rotation. Equivalently, the motion of the rotor is a pure rotation with respect to an inertial frame.
- The rotor/gear inertia is symmetric about the rotor axis of rotation so that the gravitational potential of the system and also the velocity of the rotor center of mass are both independent of the rotor position.

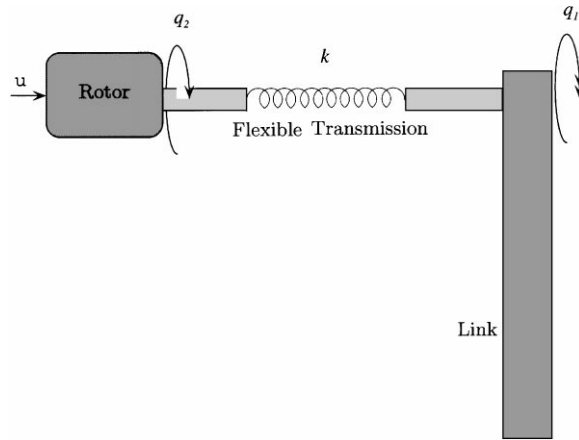


Fig. 2. A one rigid-link flexible-joint illustrative example.

Under the above assumptions, an n rigid-link multibody system with revolute joints actuated by rotors and with elasticity of the joints modeled as linear torsional springs, is suitably described by the following $2n$ -dimensional set of differential equations [16]:

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) + K(\mathbf{q}_1 - \mathbf{q}_2) = 0, \tag{29}$$

$$J\ddot{\mathbf{q}}_2 - K(\mathbf{q}_1 - \mathbf{q}_2) = \mathbf{u}, \tag{30}$$

where the vectors $\mathbf{q}_1 \in \mathbb{R}^n$ and $\mathbf{q}_2 \in \mathbb{R}^n$ represent the link angles and rotor angles, respectively, $D(\mathbf{q}_1) \in \mathbb{R}^{n \times n}$ is the inertia matrix for the rigid links, J is a diagonal matrix of rotor inertias, assumed constant, reflected to the link side of the transmission elements, $C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1$ represent the Coriolis and centrifugal terms, $\mathbf{g}(\mathbf{q}_1)$ represents the gravitational terms, and K is a diagonal matrix representing the linear joint stiffness.

The control problem under consideration consists of a tracking problem in which it is desired that the generalized coordinate vector \mathbf{q}_1 , which corresponds to the link coordinates, follow a time-varying smooth and bounded desired trajectory vector $\mathbf{q}_d(t)$ so that $\lim_{t \rightarrow \infty} [\mathbf{q}_d(t) - \mathbf{q}_1(t)] = 0$ while all other signals of the system remain bounded. The difficulty of this non-linear control problem is in the fact that active control could only be applied to the generalized coordinates \mathbf{q}_2 while the control variables of interest are the generalized coordinates \mathbf{q}_1 which are indirectly controlled, as clearly seen in Eqs. (29) and (30), through the dynamics expression $K(\mathbf{q}_1 - \mathbf{q}_2)$. We will show that by exploiting the composite control technique [15, 17, 18] in the context of the integral manifold approach discussed earlier in this paper, we solve the proposed control problem and we show that exponential stability of the equilibrium $[\mathbf{q}_d(t) - \mathbf{q}_1(t)] = 0$, equivalently exponential tracking of the desired trajectory, will be achieved for the known parameter case.

Related work which exploited the method of integral manifolds for the control of flexible joint robot manipulators includes that of [19–21]. The latter references, though, were more concerned with the analysis and control of a reduced order system restricted to the integral manifold and furthermore did not provide a stability analysis of the complete system which includes the off-manifold components. In this paper, we address the latter issues, and present a comprehensive analysis and control law design methodology based on the idea of integral manifolds. In addition, we explicitly characterize the range of joint stiffness, in terms of other system parameters and controller gains, for which we have exponential stability. Consequently, the range of stiffness that guarantees exponential stability could be optimized by altering the system parameters and/or by active control.

Note that the related control problem of requiring the motor generalized coordinates vector \mathbf{q}_2 to follow the desired trajectory $\mathbf{q}_d(t)$ is conceptually easy. This is because the dynamic equations (29)–(30) define a passive [22] mapping $\mathbf{u} \rightarrow \dot{\mathbf{q}}_2$ (input \rightarrow rotor velocity) a consequence of which is that a wealth of control results in the control literature including in particular passivity-based control laws could be directly used to control the rotor motion, while considering joint flexibility as unmodeled dynamics. But this would result in an accuracy problem since the control of link motion would become open loop which is not acceptable for precision motion control applications.

We propose a composite control law given by the following expression:

$$\mathbf{u} = \mathbf{u}_s(\mathbf{q}_1, \dot{\mathbf{q}}_1, t, \varepsilon) + \mathbf{u}_f(\dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2), \quad (31)$$

where \mathbf{u}_s is a slow controller that will be designed based on the integral manifold approach, and \mathbf{u}_f is a fast controller that will be chosen as

$$\mathbf{u}_f = K_v(\dot{\mathbf{q}}_1 - \dot{\mathbf{q}}_2), \quad (32)$$

where K_v is a constant diagonal matrix. This particular choice of the fast controller will be justified later in the paper. Note that in the limit as the joint stiffness K tends to infinity, we recover the well-known n -dimensional rigid-link model of multibody systems [16]

$$M(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{u}, \quad (33)$$

where $M(\mathbf{q}_1) = D(\mathbf{q}_1) + J$.

We will demonstrate an integral manifold control design \mathbf{u}_s that makes the dynamics of the reduced flexible system on the integral manifold coincide with those of the corresponding rigid system. Furthermore, exact computable expressions for the integral manifold will be given. We then present a detailed stability analysis and show that we can always compute an explicit range of stiffness for which tracking stability is insured for any finite initial conditions.

3.1. Integral manifold and multibody systems with rigid links and flexible joints

In order to transform the equations of motion (29) and (30) into a singularly perturbed form, we define the variable $\mathbf{z} = K(\mathbf{q}_2 - \mathbf{q}_1)$, and we assume that K is $O(1/\varepsilon^2)$, and K_v is $O(1/\varepsilon)$ so that we may write $K = K_1/\varepsilon^2$, $K_v = K_2/\varepsilon$, where K_1, K_2 are $O(1)$. By substituting the composite control law (31) in Eqs. (29) and (30), we obtain the singularly perturbed system

$$D(\mathbf{q}_1)\ddot{\mathbf{q}}_1 + C(\mathbf{q}_1, \dot{\mathbf{q}}_1)\dot{\mathbf{q}}_1 + \mathbf{g}(\mathbf{q}_1) = \mathbf{z}, \quad (34)$$

$$\varepsilon^2 J\ddot{\mathbf{z}} + \varepsilon K_2 \dot{\mathbf{z}} + K_1 \mathbf{z} = K_1(\mathbf{u}_s - J\ddot{\mathbf{q}}_1). \quad (35)$$

We introduce the following variables where Λ is a diagonal constant matrix with positive entries:

$$\tilde{\mathbf{q}}_1 = \mathbf{q}_1 - \mathbf{q}_d,$$

$$\mathbf{v} = \dot{\mathbf{q}}_d - \Lambda \tilde{\mathbf{q}}_1,$$

$$\mathbf{r} = \dot{\mathbf{q}}_1 - \mathbf{v} = \dot{\tilde{\mathbf{q}}}_1 + \Lambda \tilde{\mathbf{q}}_1,$$

$$\mathbf{a} = \dot{\mathbf{v}}.$$

After some algebra, systems (34), (35) can equivalently be written as

$$\dot{\mathbf{x}} = \mathbf{a}_1(t, \mathbf{x}) + \bar{A}_1(t, \mathbf{x})\mathbf{w}, \quad (36)$$

$$\varepsilon \dot{\mathbf{w}} = \mathbf{a}_2(t, \mathbf{x}) + \bar{A}_2(t, \mathbf{x})\mathbf{w} + B\mathbf{u}_s(t, \mathbf{x}, \varepsilon), \quad (37)$$

where

$$\mathbf{x} = \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \mathbf{r} \end{bmatrix} = \mathcal{T} \begin{bmatrix} \tilde{\mathbf{q}}_1 \\ \dot{\tilde{\mathbf{q}}}_1 \end{bmatrix} \in \mathbb{R}^{2n}$$

with the non-singular linear transformation \mathcal{T} ,

$$\mathcal{T} = \begin{bmatrix} I_n & 0_n \\ \Lambda & I_n \end{bmatrix},$$

$$\mathbf{w} = \begin{bmatrix} \mathbf{z} \\ \varepsilon \dot{\mathbf{z}} \end{bmatrix},$$

$$\mathbf{a}_1(t, \mathbf{x}) = \mathbf{a}_1(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d, \ddot{\mathbf{q}}_d) = \begin{bmatrix} \mathbf{r} - \Lambda \tilde{\mathbf{q}}_1 \\ -D^{-1} [D\mathbf{a} + C(\mathbf{r} + \mathbf{v}) + \mathbf{g}] \end{bmatrix} := \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \end{bmatrix} \in \mathbb{R}^{2n},$$

$$\bar{A}_1(t, \mathbf{x}) = \bar{A}_1(\mathbf{x}, \mathbf{q}_d) = \begin{bmatrix} 0_n & 0_n \\ D^{-1} & 0_n \end{bmatrix} := \begin{bmatrix} 0_n & 0_n \\ \mathbf{E}_{11} & 0_n \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{38}$$

$$\mathbf{a}_2(t, \mathbf{x}) = \mathbf{a}_2(\mathbf{x}, \mathbf{q}_d, \dot{\mathbf{q}}_d) = \begin{bmatrix} 0_n \\ J^{-1} K_1 J D^{-1} [C(\mathbf{r} + \mathbf{v}) + \mathbf{g}] \end{bmatrix} := \begin{bmatrix} 0 \\ \mathbf{e}_2 \end{bmatrix} \in \mathbb{R}^{2n},$$

$$\bar{A}_2(t, \mathbf{x}) = \bar{A}_2(\mathbf{x}, \mathbf{q}_d) = \begin{bmatrix} 0_n & I_n \\ -J^{-1} K_1 (I_n + J D^{-1}) & -J^{-1} K_2 \end{bmatrix} := \begin{bmatrix} 0_n & I_n \\ \mathbf{E}_{22} & \mathbf{E}_{21} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \tag{39}$$

$$B = \begin{bmatrix} 0_n \\ J^{-1} K_1 \end{bmatrix} := \begin{bmatrix} 0_n \\ \mathbf{F} \end{bmatrix} \in \mathbb{R}^{2n \times n}.$$

Denote by \mathbf{B}_x and \mathbf{B}_w compact balls centered at zero and with any fixed size. We now verify the assumptions of Lemma 2.1 for systems (36), (37) with \mathbf{x} and \mathbf{w} to \mathbf{B}_x and \mathbf{B}_w , respectively.

- \mathcal{A}_1 : The algebraic equation obtained by setting $\varepsilon = 0$ in Eq. (37) has the well-defined isolated solution

$$\mathbf{w} = -\bar{A}_2^{-1}(t, \mathbf{x})[\mathbf{a}_2(t, \mathbf{x}) + B\mathbf{u}_s(t, \mathbf{x}, \varepsilon = 0)] = -\bar{A}_2^{-1}(t, \mathbf{x})[\mathbf{a}_2(t, \mathbf{x}) + B\mathbf{u}_0],$$

where

$$\mathbf{u}_0 = \mathbf{u}_s(t, \mathbf{x}, 0).$$

In fact,

$$\bar{A}_2^{-1}(t, \mathbf{x}) = \begin{bmatrix} -\mathbf{E}_{22}^{-1} \mathbf{E}_{21} & \mathbf{E}_{22}^{-1} \\ I_n & 0_n \end{bmatrix} = \begin{bmatrix} -(I_n + J D^{-1})^{-1} K_1^{-1} K_2 & -(I_n + J D^{-1})^{-1} K_1^{-1} J \\ I_n & 0_n \end{bmatrix}.$$

Hence,

$$\mathbf{w} = \begin{bmatrix} -\mathbf{E}_{22}^{-1}(\mathbf{e}_2 + \mathbf{F}\mathbf{u}_0) \\ 0 \end{bmatrix} := h_0(t, \mathbf{x}, \mathbf{u}_0) \forall t \in \mathbb{R}, \quad \forall \mathbf{x} \in \mathbf{B}_x. \tag{40}$$

- \mathcal{A}_2 : When \mathbf{u}_s is smooth, the right-hand side of systems (36), (37) and the function $h_0(t, \mathbf{x}, \mathbf{u}_0)$ are smooth functions $\forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_x, \forall \mathbf{w} \in \mathbf{B}_w, \forall \varepsilon \in [0, \varepsilon_0]$ for a small ε_0 .
- \mathcal{A}_3 : The matrix $Z(t, \mathbf{x}) = \bar{A}_2(t, \mathbf{x})$ is non-singular $\forall t \in \mathbb{R}, \forall \mathbf{x} \in \mathbf{B}_x$ as shown above.

Since \mathcal{A}_1 – \mathcal{A}_3 are satisfied, then from Lemma 2.1 we conclude that there exists $\varepsilon_1 \leq \varepsilon_0$ such that $\forall \varepsilon \in [0, \varepsilon_1)$, $\forall \mathbf{x} \in \mathbf{B}_x$, and $\forall \mathbf{w} \in \mathbf{B}_w$, the $4n$ -dimensional flexible joint systems (36), (37) has a local $2n$ -dimensional integral manifold $\mathcal{M}_\varepsilon: \mathbf{w} = h(t, \mathbf{x}, \mathbf{u}_s, \varepsilon)$, and $h(t, \mathbf{x}, \mathbf{u}_s, \varepsilon)$ satisfies

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \left\{ \frac{\partial h}{\partial \mathbf{x}} + \frac{\partial h}{\partial \mathbf{u}_s} \frac{\partial \mathbf{u}_s}{\partial \mathbf{x}} \right\} (\mathbf{a}_1 + \bar{A}_1 h) = \mathbf{a}_2 + \bar{A}_2 h + B\mathbf{u}_s. \tag{41}$$

As $\varepsilon \rightarrow 0$, $\mathcal{M}_\varepsilon \rightarrow \mathcal{M}_0: \mathbf{w} = h_0(t, \mathbf{x}, \mathbf{u}_0)$. On the manifold, the flow is governed by the following $2n$ -dimensional system which is referred to as the reduced flexible system;

$$\dot{\mathbf{x}} = \mathbf{a}_1(t, \mathbf{x}) + \bar{A}_1(t, \mathbf{x})h(t, \mathbf{x}, \mathbf{u}_s, \varepsilon). \tag{42}$$

Fact 3.2. When $\varepsilon \rightarrow 0$, the reduced flexible system converges to

$$\dot{\mathbf{x}} = \mathbf{a}_1(t, \mathbf{x}) + \bar{A}_1(t, \mathbf{x})h_0(t, \mathbf{x}, \mathbf{u}_0) \tag{43}$$

which coincides with the rigid model given by Eq. (33).

Proof. See Appendix A. \square

3.2. Exact calculation of the integral manifold by asymptotic expansion

Let us expand \mathbf{u}_s in power of ε

$$\mathbf{u}_s(t, \mathbf{x}, \varepsilon) = \mathbf{u}_0(t, \mathbf{x}) + \varepsilon \mathbf{u}_1(t, \mathbf{x}) + \varepsilon^2 \mathbf{u}_2(t, \mathbf{x}) + \dots \tag{44}$$

Similarly, we expand the manifold function $h(t, \mathbf{x}, \varepsilon)$ in power of ε

$$h(t, \mathbf{x}, \mathbf{u}_s, \varepsilon) = h_0(t, \mathbf{x}, \mathbf{u}_0) + \varepsilon h_1(t, \mathbf{x}, \mathbf{u}_0, \mathbf{u}_1) + \varepsilon^2 h_2(t, \mathbf{x}, \mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2) + \dots \tag{45}$$

Following the procedure in Appendix B, we substitute Eqs. (45) and (44) in the manifold condition (41), and obtain

$$\varepsilon \{ \dot{h}_0 + \varepsilon \dot{h}_1 + \dots \} = \mathbf{a}_2 + \bar{A}_2 \{ h_0 + \varepsilon h_1 + \dots \} + B \{ \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \dots \}, \tag{46}$$

where \dot{h}_0 , etc., denote the total derivative computed using the procedure of approximation of integral manifolds outlined in Appendix B. Equating coefficients of equal power of ε , we obtain the following sequence of equations which solve for $h_i, i \geq 0$:

$$\varepsilon^0: 0 = \mathbf{a}_2 + \bar{A}_2 h_0 + B\mathbf{u}_0, \tag{47}$$

$$\varepsilon^1: \dot{h}_0 = \bar{A}_2 h_1 + B\mathbf{u}_1, \tag{48}$$

$$\varepsilon^k: \dot{h}_{k-1} = \bar{A}_2 h_k + B\mathbf{u}_k, \quad k \geq 2. \tag{49}$$

Eq. (47) confirms with the fact from Eq. (40) that

$$h_0 = -\bar{A}_2^{-1}(\mathbf{a}_2 + B\mathbf{u}_0) = \begin{bmatrix} -\mathbf{E}_{22}^{-1}(\mathbf{e}_2 + \mathbf{F}\mathbf{u}_0) \\ 0 \end{bmatrix} := \begin{bmatrix} \bar{h}_0 \\ 0 \end{bmatrix}. \tag{50}$$

Since the reduced flexible system corresponds, as $\varepsilon \rightarrow 0$, to the rigid system as shown in Fact 3.2, it is reasonable to choose \mathbf{u}_0 based on the rigid model. Hence, h_0 is completely defined by Eq. (47). We now choose h_1 and \mathbf{u}_1 based on Eq. (48). Once \mathbf{u}_1 is chosen, h_1 is completely determined. This process is repeated until all the desired h_i are determined. Let us consider Eq. (48), that is,

$$\begin{bmatrix} \dot{\bar{h}}_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ \mathbf{E}_{22} & \mathbf{E}_{21} \end{bmatrix} \begin{bmatrix} \bar{h}_1 \\ \bar{h}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{F}\mathbf{u}_1 \end{bmatrix} = \begin{bmatrix} \bar{h}_1 \\ \mathbf{E}_{22}\bar{h}_1 + \mathbf{E}_{21}\bar{h}_1 + \mathbf{F}\mathbf{u}_1 \end{bmatrix}.$$

Choosing $\mathbf{u}_1 = -\mathbf{F}^{-1}\mathbf{E}_{21}\dot{\bar{h}}_0$ results in

$$h_1 = \begin{bmatrix} \bar{h}_1 \\ \bar{h}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{\bar{h}}_0 \end{bmatrix}.$$

Similarly, consider from Eq. (49) for $k = 1$, $\dot{h}_1 = \bar{A}_2 h_2 + \mathbf{B}\mathbf{u}_2$, that is,

$$\begin{bmatrix} 0 \\ \dot{\bar{h}}_0 \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ \mathbf{E}_{22} & \mathbf{E}_{21} \end{bmatrix} \begin{bmatrix} \bar{h}_2 \\ \bar{h}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{F}\mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} \bar{h}_2 \\ \mathbf{E}_{22}\bar{h}_2 + \mathbf{E}_{21}\bar{h}_2 + \mathbf{F}\mathbf{u}_2 \end{bmatrix}.$$

Choosing $\mathbf{u}_2 = \mathbf{F}^{-1}\dot{\bar{h}}_1$ results in

$$h_2 = \begin{bmatrix} \bar{h}_2 \\ \bar{h}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Finally, consider Eq. (49) for $k \geq 2$. Choosing $\mathbf{u}_k = 0$ results in $h_k = 0$ for $k \geq 2$.

To summarize, if the control \mathbf{u}_s is chosen such that

$$\mathbf{u}_s = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2, \tag{51}$$

where \mathbf{u}_0 is designed based on the rigid system (43) or equivalently Eq. (33), $\mathbf{u}_1 = -\mathbf{F}^{-1}\mathbf{E}_{21}\dot{\bar{h}}_0$, $\mathbf{u}_2 = \mathbf{F}^{-1}\dot{\bar{h}}_1$, then the manifold function is given by

$$h = h_0 + \varepsilon h_1 = \begin{bmatrix} \bar{h}_0 \\ \varepsilon\dot{\bar{h}}_0 \end{bmatrix}, \tag{52}$$

where h_0 is as given by Eq. (50),

Remark 3.2. Note from Appendix B that the symbols $\dot{\bar{h}}_0$ and $\dot{\bar{h}}_1$ are defined in the following manner. First,

$$\dot{h}_0 = \frac{\partial h_0}{\partial t} + \frac{\partial h_0}{\partial \mathbf{x}} [\mathbf{a}_1 + \bar{A}_1 h_0].$$

Since

$$h_0 = \begin{bmatrix} \bar{h}_0 \\ 0 \end{bmatrix},$$

we conclude

$$\dot{\bar{h}}_0 = \frac{\partial \bar{h}_0}{\partial t} + \frac{\partial \bar{h}_0}{\partial \mathbf{x}} [\mathbf{a}_1 + \bar{A}_1 h_0].$$

Furthermore,

$$\dot{h}_1 = \frac{\partial h_1}{\partial t} + \frac{\partial h_0}{\partial \mathbf{x}} [\bar{A}_1 h_1] + \frac{\partial h_1}{\partial \mathbf{x}} [\mathbf{a}_1 + \bar{A}_1 h_0] = \frac{\partial h_1}{\partial t} + \frac{\partial h_1}{\partial \mathbf{x}} [\mathbf{a}_1 + \bar{A}_1 h_0] \quad (53)$$

since $\bar{A}_1 h_1 = 0$. From the expression

$$h_1 = \begin{bmatrix} \bar{h}_1 \\ \bar{h}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{h}_0 \end{bmatrix},$$

it follows using Eq. (53) that

$$\dot{\bar{h}} = \frac{\partial \dot{h}_0}{\partial t} + \frac{\partial \dot{h}_0}{\partial \mathbf{x}} [\mathbf{a}_1 + \bar{A}_1 h_0].$$

Remark 3.3. From Eq. (52) note that the integral manifold \mathcal{M}_ε is completely determined by h_0 . This is due to the particular choice of the control law (51) and does not hold for general non-linear singularly perturbed systems. Note also that on \mathcal{M}_ε , the dynamics of the reduced flexible system coincide with those of the rigid system. To see this, substitute Eq. (52) in Eq. (42);

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{a}_1(t, \mathbf{x}) + \bar{A}_1(t, \mathbf{x})(h_0 + \varepsilon h_1) = \mathbf{a}_1 + \bar{A}_1(t, \mathbf{x})h_0 + \begin{bmatrix} 0_n & 0_n \\ \mathbf{E}_{11} & 0_n \end{bmatrix} \begin{bmatrix} 0 \\ \varepsilon \dot{h}_0 \end{bmatrix} \\ &= \mathbf{a}_1 + \bar{A}_1(t, \mathbf{x})h_0. \end{aligned} \quad (54)$$

An immediate consequence of this fact is that any control law that stabilizes the rigid system would stabilize the reduced flexible system.

3.3. Stability analysis

In this section we follow the steps of the composite Lyapunov method which lead us to the estimation of the range of ε as was done in Theorem 2.1. Define $\eta = \mathbf{w} - h$, and let the compact ball \mathbf{B}_η , centered at zero, denote the domain of η for which $\mathbf{x} \in \mathbf{B}_\mathbf{x}$ and $\mathbf{w} \in \mathbf{B}_\mathbf{w}$. Similar to the representations (8), (9), the flexible joint systems (30), (37) can be represented on and off the manifold as

$$\dot{\mathbf{x}} = \mathbf{a}_1(t, \mathbf{x}) + \bar{A}_1(t, \mathbf{x})h_0(t, \mathbf{x}) + \bar{A}_1(t, \mathbf{x})\eta, \quad (55)$$

$$\varepsilon \dot{\eta} = \bar{A}_2(t, \mathbf{x})\eta - \varepsilon \frac{\partial h}{\partial \mathbf{x}} \bar{A}_1(t, \mathbf{x})\eta, \quad (56)$$

with $F(t, \mathbf{x}, \eta) = \mathbf{a}_1 + \bar{A}_1 h_0 + \bar{A}_1 \eta + \varepsilon \bar{A}_1 h_1 = \mathbf{a}_1 + \bar{A}_1 h_0 + \bar{A}_1 \eta$,

$$\begin{aligned} G(t, \mathbf{x}, \eta, \varepsilon) &= -\bar{A}_2 \eta + \mathbf{a}_2 + \bar{A}_2(h_0 + \varepsilon h_1) + \bar{A}_2 \eta + B(\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2) - \mathbf{a}_2 - \bar{A}_2(h_0 + \varepsilon h_1) \\ &\quad - B(\mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2) - \varepsilon \frac{\partial h}{\partial \mathbf{x}} [\mathbf{a}_1 + \bar{A}_1(h_0 + \varepsilon h_1) + \bar{A}_1 \eta - \mathbf{a}_1 - \bar{A}_1(h_0 + \varepsilon h_1)] \\ &= -\varepsilon \frac{\partial h}{\partial \mathbf{x}} \bar{A}_1 \eta. \end{aligned}$$

To make the stability analysis more concrete, let us choose \mathbf{u}_0 as the non-adaptive algorithm of Slotine and Li [23], namely,

$$\mathbf{u}_0 = (J + D)\mathbf{a} + C\mathbf{v} + \mathbf{g} - \mathbf{K}_D\mathbf{r}, \tag{57}$$

where K_D is a diagonal matrix with positive constant entries. Substituting the expression of \mathbf{u}_0 in Eqs. (55) and (56), it can be shown, after some algebra, that the flexible joint system is represented by

$$\dot{\mathbf{x}} = \mathbf{E}_1\mathbf{x} + \mathbf{E}_3\eta, \tag{58}$$

$$\varepsilon\dot{\eta} = \mathbf{E}_2\eta - \varepsilon \frac{\partial h}{\partial \mathbf{x}} \mathbf{E}_3\eta, \tag{59}$$

where

$$\mathbf{E}_1 = \begin{bmatrix} -\Lambda & I_n \\ 0_n & -(D + J)^{-1}[C + K_D] \end{bmatrix}, \quad \mathbf{E}_2 = \bar{A}_2 \quad (\text{Eq. (39)}), \quad \mathbf{E}_3 = \bar{A}_1 \quad (\text{Eq. (38)}).$$

Note that Eq. (58) with $\eta = 0$ represents the rigid system with the control \mathbf{u}_0 given by Eq. (57). Now let us choose the following Lyapunov function candidate:

$$V(t, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T P_x \mathbf{x}, \quad P_x = \begin{bmatrix} 2\Lambda K_D & 0_n \\ 0_n & (D + J) \end{bmatrix}.$$

It can be shown that the time derivative of V along the solution trajectories of the rigid system is

$$[\nabla_x V]^T \mathbf{E}_1 \mathbf{x} = -\mathbf{x}^T R \mathbf{x}, \quad R = \begin{bmatrix} 2\Lambda^T K_D \Lambda & -\Lambda^T K_D \\ -K_D^T \Lambda & K_D \end{bmatrix}$$

and that R is a positive-definite matrix. Consequently, the time derivative of V along the solution trajectories of the flexible systems (58), (59) is

$$\dot{V} = \frac{\partial V}{\partial t} + [\nabla_x V]^T \mathbf{E}_1 \mathbf{x} + [\nabla_x V]^T \mathbf{E}_3 \eta = -\mathbf{x}^T R \mathbf{x} + \mathbf{x}^T R_1 \eta \leq -\tau_2 \|\mathbf{x}\|^2 + \tau_3 \|\mathbf{x}\| \|\eta\|, \tag{60}$$

where

$$\tau_2 = \lambda_{\min}[R] \tag{61}$$

and

$$\tau_3 = \|R_1\|, \quad R_1 = \begin{bmatrix} 0_n & 0_n \\ (D + J)D^{-1} & 0_n \end{bmatrix}. \tag{62}$$

At this point, we verify assumption \mathcal{A}_4 and check under what conditions the eigenvalues of $\mathbf{E}_2 = \bar{A}_2$ have negative real parts. Consider the auxiliary differential equation

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\dot{\mathbf{p}}} \end{bmatrix} = \bar{A}_2 \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} 0_n & I_n \\ -J^{-1}K_1(I_n + JD^{-1}) & -J^{-1}K_2 \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}, \quad \mathbf{p} \in \mathbb{R}^n,$$

or, equivalently,

$$K_2^{-1}J\ddot{\mathbf{p}} + \dot{\mathbf{p}} + (K_2^{-1}K_1 + K_2^{-1}K_1JD^{-1})\mathbf{p} = 0. \tag{63}$$

This second-order vector differential equation (with initial conditions $\mathbf{p}(0) = \mathbf{p}_0, \dot{\mathbf{p}}(0) = \dot{\mathbf{p}}_0$) is asymptotically stable for frozen \mathbf{x} (i.e. eigenvalues of \bar{A}_2 have strictly negative real parts for frozen \mathbf{x}) if (see [24, 25]), $K_2^{-1}J > 0$ ($> 0 =$ positive definite), and $K_2^{-1}K_1 + K_2^{-1}K_1JD^{-1} > 0$. With the choice

$$K_2 := K_1J, \tag{64}$$

Eq. (63) becomes

$$J^{-1}K_1^{-1}J\ddot{\mathbf{p}} + \dot{\mathbf{p}} + (J^{-1} + D^{-1})\mathbf{p} = 0.$$

Since D and D^{-1} are bounded and $D > 0$, it follows that $D^{-1} > 0$. J^{-1} is a diagonal matrix with positive diagonal elements, hence positive definite. The sum of two positive-definite matrices is positive definite, hence $(J^{-1} + D^{-1}) > 0$. Since K_1 is diagonal and positive definite, the matrix product $J^{-1}K_1^{-1}J$ is diagonal and positive definite. To summarize, the choice of K_2 in Eq. (64) ensures that the eigenvalues of \bar{A}_2 have strictly negative real parts.

Since $\mathbf{E}_2 = Z(t, \mathbf{x})$ now satisfies all the assumption of Lemma 2.3 when K_2 is chosen as in Eq. (64), then we can choose the Lyapunov function $W(t, \mathbf{x}, \eta) = \eta^T P(t, \mathbf{x})\eta$. From Lemma 2.3, we conclude that

$$[\nabla_{\eta}W(t, \mathbf{x}, \eta)]^T \left(\frac{1}{\varepsilon} \mathbf{E}_2 \eta \right) + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon)\eta \leq \left(-\frac{1}{\varepsilon} + \tau_1 \right) \|\eta\|^2.$$

Consequently, the time derivative of W along the solution trajectories of the flexible systems (58), (59) is

$$\begin{aligned} \dot{W} &= [\nabla_{\eta}W]^T \left(\frac{1}{\varepsilon} \mathbf{E}_2 \eta \right) + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon)\eta + [\nabla_{\eta}W]^T \frac{1}{\varepsilon} \left(-\varepsilon \frac{\partial h}{\partial \mathbf{x}} \mathbf{E}_3 \eta \right) \\ &= [\nabla_{\eta}W]^T \left(\frac{1}{\varepsilon} \mathbf{E}_2 \eta \right) + \eta^T \dot{P}(t, \mathbf{x}, \eta, \varepsilon)\eta - 2\eta^T P(t, \mathbf{x}) \frac{\partial h}{\partial \mathbf{x}} \mathbf{E}_3 \eta \\ &\leq \left(-\frac{1}{\varepsilon} + \tau_1 \right) \|\eta\|^2 + \tau_4 \|\eta\|^2, \end{aligned} \tag{65}$$

where

$$\tau_4 := 2 \left\| P(t, \mathbf{x}) \left[\frac{\partial h_0}{\partial \mathbf{x}} + \varepsilon \frac{\partial h_1}{\partial \mathbf{x}} \right] \mathbf{E}_3 \right\|. \tag{66}$$

We now choose the composite Lyapunov function

$$\mathcal{V}(t, \mathbf{x}, \eta) = d_1 V(t, \mathbf{x}) + d_2 W(t, \mathbf{x}), \quad d_1, d_2 > 0.$$

Using Eqs. (60) and (65), the time derivative of \mathcal{V} along the solution trajectories of the flexible joint systems (58), (59) is

$$\begin{aligned} \dot{\mathcal{V}}(t, \mathbf{x}, \eta) &\leq -d_1 \tau_2 \|\mathbf{x}\|^2 + d_1 \tau_3 \|\mathbf{x}\| \|\eta\| - \frac{d_2}{\varepsilon} \|\eta\|^2 + d_2(\tau_1 + \tau_4) \|\eta\|^2 \\ &= -[\|\mathbf{x}\| \|\eta\|] \begin{bmatrix} d_1 \tau_2 & -\frac{d_1}{2} \tau_3 \\ -\frac{d_1}{2} \tau_3 & \frac{d_2}{\varepsilon} - d_2(\tau_1 + \tau_4) \end{bmatrix} \begin{bmatrix} \|\mathbf{x}\| \\ \|\eta\| \end{bmatrix} \end{aligned}$$

which is negative definite when $\varepsilon < \tau_2/(\tau_2(\tau_1 + \tau_4) + (d_1/4d_2)\tau_3^2)$.

We therefore have the following theorem which is a specialized version of Theorem 2.1.

Theorem 3.2. *Given the flexible joint systems (36), (37) with control law \mathbf{u}_s satisfying Eq. (51), and K_2 satisfying Eq. (64), then there exists an upper bound of ε given by*

$$\varepsilon_d^i = \frac{\tau_2}{\tau_2(\tau_1 + \tau_4) + (d_1/4d_2)\tau_3^2} \tag{67}$$

such that $\forall \varepsilon \in [0, \varepsilon_d^i], \forall (t_0, \mathbf{x}(t_0), \eta(t_0)) \in \Omega_{\mathcal{V}}$ where

$$\Omega_{\mathcal{V}} = \{ \mathbf{x} \in \mathbf{B}_x, \eta \in \mathbf{B}_\eta : \mathcal{V}(t, \mathbf{x}, \eta) \leq \min[d_1 c_v, d_2 c_w] \}, \tag{68}$$

the equilibrium $\mathbf{x} = 0, \eta = 0$ (i.e. $\mathbf{w} = h_0 + \varepsilon h_1$) is exponentially stable, d_1, d_2 are any positive real constants, $c_v, c_w, \tau_1, \tau_2, \tau_3,$ and τ_4 are given by Eqs. (22), (23), (17), (62), and (66), respectively.

Recall that there is no restriction on the size of the compact balls \mathbf{B}_x and \mathbf{B}_η . Hence, Theorem 3.2 is valid for all finite initial conditions. Note though that the larger the size of the balls is, the smaller the resulting ε_d^i becomes as the expression of ε_d^i in Eq. (67) suggests.

The main conclusion of this section can be stated in the following corollary:

Corollary 3.1. *Given the rigid-link-flexible-joint multibody systems (29), (30), the composite control law (31), namely, $\mathbf{u} = \mathbf{u}_s + \mathbf{u}_f$ where the fast controller \mathbf{u}_f is given by Eq. (32) and the slow controller \mathbf{u}_s is designed using the integral manifold approach proposed in this paper and is given by Eq. (51), insures that the link generalized coordinate vector $\mathbf{q}_1(t)$ exponentially track any smooth and bounded desired trajectory vector $\mathbf{q}_d(t)$. This exponential stability is guaranteed for a range of stiffness of the joints that is inversely proportional to ε_d^i given by Eq. (67).*

4. Conclusion

In this paper we have first reviewed the existence and properties of attractive integral manifolds of singularly perturbed non-linear differential equations and discussed and outlined the proof. We then exploited the composite Lyapunov method to establish stability properties of the full singularly perturbed system by combining one Lyapunov function corresponding to the dynamics on the manifold and a second Lyapunov function corresponding to the dynamics of the manifold. As a by-product of this Lyapunov stability analysis, an explicit range of the small parameter in terms of the other system’s parameters is obtained. In the context of control system design, the controller ought to be designed to insure the existence of an attractive manifold as well as the stability of the equilibrium of the dynamics when restricted to the manifold. Hence, the manifold itself is also dependent on the choice of the controller. It follows that there is usually a lot of freedom in the design of an integral manifold based control law.

The method of integral manifolds was applied to multibody systems with rigid links and flexible joints which are modeled as a singularly perturbed system where the smaller parameter is taken as inversely proportional to joint stiffness. The proposed control law, which uses only positions and velocities for feedback, is a composite controller with a fast component responsible for the attractivity of the manifold, and a slow component designed to shape the manifold and to insure the stability of the dynamics restricted to the manifold. The controller was designed in such a manner that the dynamics restricted to the manifold coincide with the dynamics of the corresponding rigid system obtained by letting the joint stiffness become very large. This is particularly important as all control laws that stabilize the rigid multibody system would stabilize the dynamics of the flexible system on the manifold. The composite Lyapunov stability method was applied and exponential stability of the equilibrium of the flexible multibody system was established and explicit range of joint stiffness was given for which the stability properties are guaranteed.

Appendix A. Proof of Fact 3.2

Substituting h_0 from Eq. (40) into the reduced flexible system (42), we obtain

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{a}_1 + \bar{A}_1 h_0 = \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \end{bmatrix} + \begin{bmatrix} 0_n & 0_n \\ \mathbf{E}_{11} & 0_n \end{bmatrix} \begin{bmatrix} -\mathbf{E}_{22}^{-1}(\mathbf{e}_2 + \mathbf{F}\mathbf{u}_0) \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\mathbf{E}_{11}\mathbf{E}_{22}^{-1}(\mathbf{e}_2 + \mathbf{F}\mathbf{u}_0) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}(I_n + JD^{-1})^{-1}K_1^{-1}J\{J^{-1}K_1JD^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) + J^{-1}K_1\mathbf{u}_0\} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ D^{-1}(I_n + JD^{-1})^{-1}JD^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) + D^{-1}(I_n + JD^{-1})^{-1}\mathbf{u}_0 \end{bmatrix}.\end{aligned}\quad (69)$$

Note that

$$D^{-1}(I_n + JD^{-1})^{-1} = D^{-1}((D + J)D^{-1})^{-1} = (D + J)^{-1}\quad (70)$$

and

$$\begin{aligned}D^{-1}(I_n + JD^{-1})^{-1}JD^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) &= D^{-1}(J(J^{-1} + D^{-1}))^{-1}JD^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) \\ &= D^{-1}(J^{-1} + D^{-1})^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) = D^{-1}\{(D - D(D + J)^{-1}D)D^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g})\} \\ &= D^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) - (D + J)^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}).\end{aligned}\quad (71)$$

Substituting Eqs. (70), (71) and the expressions for \mathbf{e}_0 and \mathbf{e}_1 , Eq. (69) becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{r} - \Lambda\tilde{\mathbf{q}}_1 \\ \mathbf{a} - (D + J)^{-1}(C(\mathbf{r} + \mathbf{v}) + \mathbf{g}) + (D + J)^{-1}\mathbf{u}_0 \end{bmatrix}$$

or equivalently the rigid model

$$(D + J)\ddot{\mathbf{q}}_1 + C\dot{\mathbf{q}} + \mathbf{g} = \mathbf{u}_0.$$

Appendix B. Formal solution of the integral manifold function

In general, it is impossible to solve for the function $h(t, \mathbf{x}, \varepsilon)$ from Eq. (6) exactly, since it is equivalent to solving the original systems (2), (3). Following Strygin and Sobelov [26], the coefficients $h_i(t, \mathbf{x})$, $i = 0, 1, 2, \dots$ in Eq. (45) are uniquely determined from the formula identity

$$\varepsilon \left\{ \sum_{i=0}^{\infty} \varepsilon^i \frac{\partial h_i}{\partial t} + \sum_{i=0}^{\infty} \varepsilon^i \frac{\partial h_i}{\partial \mathbf{x}} f \left(t, \mathbf{x}, \sum_{j=0}^{\infty} \varepsilon^j h_j, \varepsilon \right) \right\} = g \left(t, \mathbf{x}, \sum_{i=0}^{\infty} \varepsilon^i h_i, \varepsilon \right)\quad (72)$$

which is obtained by substituting Eq. (45) in the integral manifold equation (6). To formally compute the coefficients, we first write the Taylor series expansion of $f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon)$ and $g(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon)$ about $\varepsilon = 0$, that is,

$$f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon) = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots + \varepsilon^{k-1} f_{k-1} + R_f(t, \mathbf{x}, \varepsilon)\quad (73)$$

where, for example,

$$\begin{aligned}
 f_0 &= f(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon)|_{\varepsilon=0} = f(t, \mathbf{x}, h_0 + \varepsilon h_1 + \dots, \varepsilon)|_{\varepsilon=0} = f(t, \mathbf{x}, h_0(t, \mathbf{x}), 0), \\
 f_1 &= \left\{ \frac{\partial f}{\partial h} \frac{\partial h}{\partial \varepsilon} \right\}_{\varepsilon=0} + \left\{ \frac{\partial f}{\partial \varepsilon} \right\}_{\varepsilon=0} = \frac{\partial f}{\partial h}(t, \mathbf{x}, h_0(t, \mathbf{x}), 0)h_1(t, \mathbf{x}) + \frac{\partial f}{\partial \varepsilon}(t, \mathbf{x}, h_0(t, \mathbf{x}), 0), \\
 f_2 &= \frac{1}{2} \left\{ \frac{\partial}{\partial \varepsilon} \frac{\partial f}{\partial h} \frac{\partial h}{\partial \varepsilon} \right\}_{\varepsilon=0} + \frac{1}{2} \left\{ \frac{\partial^2 f}{\partial \varepsilon^2} \right\}_{\varepsilon=0} = \frac{1}{2} \left\{ \frac{\partial}{\partial \varepsilon} \frac{\partial f}{\partial h}(h_1 + 2\varepsilon h_2 + \dots) \right\}_{\varepsilon=0} + \frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0) \\
 &= \frac{1}{2} \left\{ \frac{\partial}{\partial \varepsilon} \left(\frac{\partial f}{\partial h} h_1 + 2\varepsilon \frac{\partial f}{\partial h} h_2 + \dots \right) \right\}_{\varepsilon=0} + \frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0) \\
 &= \frac{1}{2} \left\{ \frac{\partial((\partial f/\partial h)h_1)}{\partial h}(h_1 + 2\varepsilon h_2 + \dots) + 2 \frac{\partial f}{\partial h} h_2 \right. \\
 &\quad \left. + 2\varepsilon \frac{\partial((\partial f/\partial h)h_2)}{\partial h}(h_1 + 2\varepsilon h_2 + \dots) \right\}_{\varepsilon=0} + \frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0) \\
 &= \frac{\partial f}{\partial h} h_2 + \frac{1}{2} \frac{\partial((\partial f/\partial h)h_1)}{\partial h}(t, \mathbf{x}, h_0, h_1) + \frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0).
 \end{aligned}$$

$R_f(t, \mathbf{x}, \varepsilon)$ is the remainder of the Taylor series. Similarly,

$$g(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon) = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \dots + \varepsilon^{k-1} g_{k-1} + R_g(t, \mathbf{x}, \varepsilon) \tag{74}$$

where

$$\begin{aligned}
 g_0 &= g(t, \mathbf{x}, h(t, \mathbf{x}, \varepsilon), \varepsilon)|_{\varepsilon=0} = g(t, \mathbf{x}, h_0(t, \mathbf{x}), 0), \\
 g_1 &= \left\{ \frac{\partial g}{\partial h} \frac{\partial h}{\partial \varepsilon} \right\}_{\varepsilon=0} + \left\{ \frac{\partial g}{\partial \varepsilon} \right\}_{\varepsilon=0} = Z(t, \mathbf{x})h_1(t, \mathbf{x}) + \frac{\partial g}{\partial \varepsilon}(t, \mathbf{x}, h_0(t, \mathbf{x}), 0), \\
 g_2 &= \frac{1}{2} \left\{ \frac{\partial}{\partial \varepsilon} \frac{\partial g}{\partial h} \frac{\partial h}{\partial \varepsilon} \right\}_{\varepsilon=0} + \frac{1}{2} \left\{ \frac{\partial^2 g}{\partial \varepsilon^2} \right\}_{\varepsilon=0} = \frac{\partial g}{\partial h} h_2 + \frac{1}{2} \frac{\partial((\partial g/\partial h)h_1)}{\partial h}(t, \mathbf{x}, h_0, h_1) + \frac{1}{2} \frac{\partial^2 g}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0) \\
 &= Z(t, \mathbf{x})h_2 + \frac{1}{2} \frac{\partial((\partial g/\partial h)h_1)}{\partial h}(t, \mathbf{x}, h_0, h_1) + \frac{1}{2} \frac{\partial^2 g}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0).
 \end{aligned}$$

In general, for $l \leq k - 1$,

$$g_l = Z(t, \mathbf{x})h_l + \bar{g}_l(t, \mathbf{x}, h_0, h_1, \dots, h_{l-1}).$$

$R_g(t, \mathbf{x}, \varepsilon)$ is the remainder of the Taylor series. Replacing Eqs. (73) and (74) in Eq. (72), we obtain

$$\begin{aligned}
 &\varepsilon \left\{ \frac{\partial h_0}{\partial t} + \varepsilon \frac{\partial h_1}{\partial t} + \dots + \varepsilon^{k-1} \frac{\partial h_{k-1}}{\partial t} + \varepsilon^k \frac{\partial h_k(t, \mathbf{x}, \varepsilon)}{\partial t} \right\} \\
 &+ \varepsilon \left\{ \frac{\partial h_0}{\partial \mathbf{x}} + \varepsilon \frac{\partial h_1}{\partial \mathbf{x}} + \dots + \varepsilon^{k-1} \frac{\partial h_{k-1}}{\partial \mathbf{x}} + \varepsilon^k \frac{\partial h_k(t, \mathbf{x}, \varepsilon)}{\partial \mathbf{x}} \right\} \\
 &\times \{ f_0 + \varepsilon f_1 + \dots + \varepsilon^{k-1} f_{k-1} + R_f(t, \mathbf{x}, \varepsilon) \} = g_0 + \varepsilon g_1 + \dots + \varepsilon^{k-1} g_{k-1} + R_g(t, \mathbf{x}, \varepsilon).
 \end{aligned}$$

Equating equal powers of ε , we obtain

$$\varepsilon^0: 0 = g_0 = g(t, \mathbf{x}, h_0(t, \mathbf{x}), 0)$$

$$\varepsilon^1: \dot{h}_0 =: \frac{\partial h_0}{\partial t} + \frac{\partial h_0}{\partial \mathbf{x}} f_0 = g_1 = Z(t, \mathbf{x}) h_1(t, \mathbf{x}) + \frac{\partial g}{\partial \varepsilon}(t, \mathbf{x}, h_0(t, \mathbf{x})). \quad (75)$$

$$\varepsilon^2: \dot{h}_1 =: \frac{\partial h_1}{\partial t} + \frac{\partial h_0}{\partial \mathbf{x}} f_1 + \frac{\partial h_1}{\partial \mathbf{x}} f_0 = g_2 = Z(t, \mathbf{x}) h_2 + \frac{1}{2} \frac{\partial((\partial g / \partial h) h_1)}{\partial h}(t, \mathbf{x}, h_0, h_1) + \frac{1}{2} \frac{\partial^2 g}{\partial \varepsilon^2}(t, \mathbf{x}, h_0, 0). \quad (76)$$

$$\begin{aligned} \varepsilon^{k-1}: \dot{h}_{k-2} &=: \frac{\partial h_{k-2}}{\partial t} + \frac{\partial h_0}{\partial \mathbf{x}} f_{k-2} + \frac{\partial h_1}{\partial \mathbf{x}} f_{k-3} + \dots + \frac{\partial h_{k-2}}{\partial \mathbf{x}} f_0 = g_{k-1} \\ &= Z(t, \mathbf{x}) h_{k-1}(t, \mathbf{x}) + \bar{g}_{k-1}(t, \mathbf{x}, h_0, h_1, \dots, h_{k-2}). \end{aligned} \quad (77)$$

Since $Z(t, \mathbf{x})$ is non-singular, we can solve uniquely for h_1 and h_i , $i = 2, 3, \dots, k-1$ from Eqs. (75)–(77).

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